IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Semisimple Lie superalgebras which are not the direct sums of simple Lie superalgebras

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1987 J. Phys. A: Math. Gen. 20 1

(http://iopscience.iop.org/0305-4470/20/1/010)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 31/05/2010 at 16:06

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **20** (1987) 1-14. Printed in the UK

Semisimple Lie superalgebras which are not the direct sums of simple Lie superalgebras†

J P Hurni

Département de Physique Théorique, Université de Genève, 1211 Genève 4, Switzerland

Received 25 September 1985, in final form 30 January 1986

Abstract. Any finite-dimensional semisimple Lie algebra (LA) is known to be realised as a direct sum of simple LA. The situation is somewhat more complicated in the Lie superalgebra (LSA) case. A theorem of Kac states how to build all the semisimple LSA. The goal of this paper is to give some explicit examples arising from this theorem, such illustrations perhaps being better understood by physicists rather than an abstract statement; and to show the relevance of the Neveu-Schwarz and Ramond superalgebras when generalising the theorem to the infinite case.

1. Introduction

It is well known that in the case of finite Lie algebras, these four statements are equivalent : L is a semisimple LA if:

$$\text{L does not contain any solvable ideal} \tag{1.1}$$

$$\text{L is the direct sum of simple LA} \tag{1.2}$$

$$\text{L has a non-degenerate Killing form} \tag{1.3}$$

$$\text{all the finite representations of L are completely reducible.} \tag{1.4}$$

Remember that a solvable Lie algebra R satisfies by definition the following property:

$$\begin{aligned} &[\mathrm{R}, \mathrm{R}] = \mathrm{R}^1 \\ &[\mathrm{R}^1, \mathrm{R}^1] = \mathrm{R}^2 \\ &\quad\vdots \\ &[\mathrm{R}^n, \mathrm{R}^n] = 0 \end{aligned}$$

where [,] denotes the traditional bracket (R is an ideal of L if $[\mathrm{L}, \mathrm{R}] \subset \mathrm{R}$).

Given the definition (1.1) it is therefore easy to understand intuitively the following theorem: let L be any Lie algebra, then $\mathrm{S} \equiv \mathrm{L}/\mathrm{R}$, where R denotes the maximal solvable ideal, is semisimple.

Clearly, this theorem does not give a very practical way of finding all the possible semisimple Lie algebras. Happily all the simple Lie algebras are classified; therefore all the semisimple LA are easily found thanks to (1.2).

† Partially supported by the Swiss National Science Foundation.

0305-4470/87/010001 + 14$02.50 © 1987 IOP Publishing Ltd

When considering LSA instead of LA, the four above-mentioned statements are pairwise inequivalent, each statement being strictly less general than its predecessor. Therefore (1.1), when considering the standard bracket $[\ ,\ \}$ for the LSA, is the only exhaustive definition of a semisimple LSA.

All the simple LSA have been classified [1, 2]; some of them have a non-degenerate Killing form, in which case it is indicated by $*$ in table A1.

Thanks to table A1 the semisimple LSA satisfying (1.2) are easy to build. A theorem states that (1.3) is the direct sum of simple LSA, each having a non-degenerate Killing form [1]. Furthermore, it is well known that only for direct sums of the $B(0, n)$ class of LSA (and the simple LA seen as trivially graded LSA) does the complete reducibility of representations hold [5]; this gives (1.4).

The semisimple LSA satisfying (1.2)–(1.4) are direct sums of simple LSA. Therefore their structures are easily understood once we know the structures of their respective simple sub-LSA.

The structure of the semisimple LSA which are not direct sums of simple ones can be recovered using a theorem due to Kac [1], thereafter called the main theorem. The structure of the resulting LSA may look somewhat unfamiliar to physicists. Therefore this paper tries to shed some light on particular aspects of the theorem by means of many examples.

In § 2 we recall some basic definitions and theorems. In § 3 we present the 'main theorem'. Section 4 is divided into five 'examples', each giving an illustration of a particular aspect of the general problem. Using the main theorem, it is possible (§ 5) to construct (infinite-dimensional) semisimple LA which are not direct sums of simple ones. An example is the semidirect sum of a loop algebra by the centreless Virasoro algebra. In the second part of § 5 we consider the extension to the infinite superalgebras case.

2. Reminder of some results

Definitions. A superalgebra $A = A_0 + A_1$ is a Z_2-graded algebra such that $X_i X_j \in A_{i+j}\ \forall X_k \in A_k$; a Lie superalgebra $G = G_0 + G_1$ is a superalgebra where the product, denoted $[\ ,\ \}$, satisfies the following properties:

$$[X, Y\} = -(-1)^{\varepsilon(X)\varepsilon(Y)}[Y, X\} \qquad \text{(graded anticommutativity)}$$

$$\sum_{\mathrm{CP}} -(-1)^{\varepsilon(X)\varepsilon(Z)}[X, [Y, Z\}\} = 0 \qquad \text{(super-Jacobi identity)}$$

where $\varepsilon(X)$ denotes the degree of X and Σ_{CP} the cyclic permutations of X, Y and Z.

A given associative superalgebra can be turned into a Lie superalgebra if we identify $[a, b\}$ with the commutator $[a, b] \equiv ab - ba$ when a and/or b is even, and with the anticommutator $\{a, b\} \equiv ab + ba$ when both a and b are odd.

Theorem. Let G be a finite-dimensional Lie superalgebra, then G contains a unique maximal solvable ideal R. Then the Lie superalgebra G/R is semisimple.

Example. Physicists are interested in the super-Poincaré algebra $P = P_0 + S$, where P_0 is the Lorentz algebra so(3, 1), and where $S \equiv P_1 + P_2$ is the so-called supersymmetry algebra (spanned by the pure supersymmetry generators Q_α, $\bar{Q}_{\dot\alpha} \in P_1$, $\alpha, \dot\alpha = 1, 2$, and the translations $T_i \in P_2$, $i = 0, \ldots, 3$ with the following commutation relations:

$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma^i)_{\alpha\dot{\alpha}} T_i$, the other brackets remaining zero). P has a consistent Z gradation, i.e. $[P_i, P_j\} \subset P_{i+j}$. Thus P is clearly non-semisimple, S being the maximal solvable ideal. We can effectively show, according to the theorem, that P/S is a Lie superalgebra (isomorphic to the Lorentz algebra) which is semisimple.

Remark. In general the factor space G/R is not simply the subspace of G that does not contain R. For example, $G \approx su(n+1/n+1)$ has a one-dimensional centre which is the maximal solvable ideal R: $G = F + R$. However the anticommutation of some odd generators give contributions to the centre, $[F, F\} = G$. Thus F is not a LSA. However it is possible to redefine the anticommutators in such a way that $[F, F\}' = F$. In that case F ($\approx G/R$) is isomorphic to the simple (thus *a fortiori* semisimple) LSA called $A(n, n)$. Strictly speaking, $su(n+1/n+1)$ is the central extension of $A(n, n)$.

Definitions. Let the G_i denote subspaces of G; $G_i \cap G_j = 0\ \forall i \neq j$ then, if $[G, G_i\} = G_{i'} + G_{i''} + G_{i'''} + \ldots$, a *graph* of G is the set of all the arrows coming from the G_i going to each $G_{i''\ldots''} \neq G_i$. A graph is generally not unique, e.g. super-Poincaré has the following graphs: $P_0 \to S$, $P_0 \to P_1 \to P_2$, and many others.

It is clearly an alternative notation for the semi-direct sum:

$$V_1 \oplus V_2 \equiv V_1 \leftarrow V_2.$$

Let V be a Z_2-graded space, $V = V_0 + V_1$, then the *superdimension* of V, sdim V, is the dimension of V_0 minus the dimension of V_1.

Proposition 1. Let $V = V_0 + V_1$ be a finite irreducible representation of the solvable LSA $G = G_0 + G_1$, then

$$\begin{array}{lll} \text{either} & \operatorname{sdim} V = 0 \text{ and } \dim V = 2^s & 0 < s \leqslant \dim G_1 \\ \text{or} & \dim V = 1 (= \operatorname{sdim} V). & \end{array}$$

Proposition 2. All irreducible representations of the solvable LSA G are one dimensional if and only if $\{G_1, G_1\} \subset [G_0, G_0]$.

Corollary. The supersymmetry algebra admits non-trivial finite irreducible representations, as $\{P_1, P_1\} = P_2 \not\subset 0 = [P_2, P_2]$; therefore any faithful irreducible representation V has sdim $V = 0$. This is a well known result [3].

3. The main theorem

Definition 1. Let $A = A_0 + A_1$ be a superalgebra, der A the Lie superalgebra of its derivations, and L a subset of der A, then A is said to be 'L-simple' if A contains no non-trivial ideals that are invariant under all derivations in L.

Definition 2. Let G be a Lie superalgebra, then der G denotes the Lie superalgebra of its derivations, and inder G ($\approx$G when G is centreless) the ideal of der G consisting of the inner derivations.

Definition 3. $\Lambda(n)$ denotes the 2^n-dimensional Grassmann algebra in n variables $\xi_1, \ldots, \xi_n$ ($\xi_i\xi_j = -\xi_j\xi_i$, $\forall i, j$). A Grassmann algebra is an associative superalgebra (we put $\varepsilon(\xi_i) = 1$).

Definition 4. der $\Lambda(n) \approx W(n)$ is the 2^n n-dimensional LSA of the derivations of $\Lambda(n)$. The basis vectors $\{D_i\}$ can be written in the following way: $\{P(\xi_1, \ldots, \xi_n) \cdot \partial_{\xi_i}\}$, where the derivations ∂_{ξ_i} defined by $\partial_{\xi_i}(\xi_j) \equiv \delta_{ij}$ are multiplied by polynomials in the ξ_i. $W(n)$ is simple for $n \geqslant 2$. Note that $W(2) \approx \mathrm{sl}(1/2)$.

Definition 5. Let A be a sub-LSA of a LSA called B. Then the *normaliser* of A in B is the subset N of B such that $[N, \mathrm{A}\} \subset \mathrm{A}$.

Lemma. Let G be a LSA and $\Lambda(n)$ a Grassmann algebra. Then $\mathrm{S} = \mathrm{G} \otimes \Lambda(n)$ is also a LSA.

Proof. A generator of S can be seen as a couple $X \otimes u$ where X is a generator of G and u a generator of $\Lambda(n)$. The generators of S have to satisfy the graded anticommutation relations and the super-Jacobi identities. The following definition of the bracket:

$$[X \otimes u, X' \otimes u'\} \equiv (-1)^{\varepsilon(u)\varepsilon(X')}([X, X'\} \otimes u \cdot u')$$

is consistent with the structure of a LSA.

We see that $X \otimes u$ is even if both X and u are even or odd, and that $X \otimes u$ is odd if X and u do not have the same parity.

Remark. $\mathrm{S} = \mathrm{G} \otimes \Lambda(n)$, $n \geqslant 1$, cannot be a semisimple LSA, as the set of generators of the form $X \otimes u$, where $u \neq 1$, span an ideal solvable of S, due to the nilpotent of the ξ_i.

To formulate the main theorem, we still need the following ingredients. Let $\mathrm{S}_1, \ldots, \mathrm{S}_r$ be finite simple Lie superalgebras, $n_1, \ldots, n_r$ be non-negative integers and $\mathrm{S} = \bigoplus_{i=1}^r \mathrm{S}_i \otimes \Lambda(n_i)$. Then

$$\mathrm{S} \equiv \mathrm{inder}\,\mathrm{S} = \bigoplus_{i=1}^{r} (\mathrm{inder}\,\mathrm{S}_i) \otimes \Lambda(n_i) \subset \mathrm{der}\,\mathrm{S} \equiv \bigoplus_{i=1}^{r} ((\mathrm{der}\,\mathrm{S}_i) \otimes \Lambda(n_i) \oplus 1 \otimes \mathrm{der}\,\Lambda(n_i)).$$

The commutation relations among generators A and B of der S are the following.

(i) According to the lemma, we already know them if both A and B belong to $\bigoplus_i \mathrm{der}\,\mathrm{S}_i \otimes \Lambda(n_i)$.

(ii) Finally the obvious action of der S on S makes der S a Z_2-graded associative algebra. The resulting Lie superalgebra is given by the natural commutators/anticommutators. In particular, if both $\mathrm{A} = 1 \otimes D$ and $\mathrm{B} = 1 \otimes D'$ belong to $1 \otimes \mathrm{der}\,\Lambda(n_i)$, then we simply have $[1 \otimes D, 1 \otimes D'\} \equiv 1 \otimes [D, D'\}$, and $[1 \otimes D, X \otimes u\} \equiv X \otimes D(u)$, where

X belongs now to der S_i and $D(u)$ denotes the action of the derivation $D \in W(n_i)$ on the element $u \in \Lambda(n_i)$; clearly $D(u)$ belongs to $\Lambda(n_i)$.

Main theorem. Let L be a subalgebra of der S containing S; we denote by L_i the set of components of elements of L in $1 \otimes \operatorname{der} \Lambda(n_i)$. Then

(a) L is semisimple if and only if $\Lambda(n_i)$ is L_i-simple for all i;
(b) all finite-dimensional semisimple LSA arise in the manner indicated;
(c) der L is the normaliser of L in der S, provided that L is semisimple.

Discussion. It is clear from the definitions of the superbracket that L, a semisimple LSA, consists essentially of the semidirect sum of two LSA: one which is S itself, i.e. $\bigoplus_i^r \operatorname{inder} S_i \otimes \Lambda(n_i)$, and another one which is essentially a sub-LSA of a direct sum of the simple LSA der $\Lambda(n_i) \approx W(n_i)$.

If some of the simple LSA S_i involved in the above construction do not coincide with the LSA of their derivations, there may be additional terms in L that do not belong to S and the $W(n_i)$, as we shall see in the next section.

The descriptions of the simple S_i and their corresponding der S_i, which are given in table A1, can be found in [1, 2].

In the following, when $S = \bigoplus_i^r \operatorname{inder} S_i \otimes \Lambda(n_i) \approx \bigoplus_i^r S_i \otimes \Lambda(n_i)$, $r = 1$, S_1 will be written as G in order to avoid confusion with the Z_2-grading of this last LSA.

Proof of the main theorem. See [1].

4. Examples

According to the main theorem, there are many possible semisimple LSA, embedding S, embedded in der S.

The main difficulty is to satisfy the point (a) of the main theorem; therefore in the first example we choose $S = G \otimes \Lambda(n)$ to be $\mathrm{sl}(1/2) \otimes \Lambda(2)$ for the following reasons: the ideals of $\Lambda(1)$ are too simple, and by choosing $\mathrm{sl}(1/2)$ we avoid a complication that we reserve for the second example.

In the first example we exhibit explicitly four inequivalent semisimple LSA, but the reader can convince himself or herself that finding the number of inequivalent semisimple LSA that can appear is quite similar in a certain sense to the classification of all the subalgebras of the simple LSA $W(2)$ in that particular case; $W(n)$ in the general case. The second example shows that there are inequivalent semisimple LSA which satisfy the point (a) of the main theorem in exactly the same fashion. This occurs when the simple LSA S_i are not isomorphic to the LSA of their derivations. The third example speaks for itself. The fourth example shows that we really have to consider all the possibilities allowed by the concise formulation of the main theorem.

In many of these examples we give, for some L, the corresponding der L that we obtain according to the point (c) of the main theorem. Note that each der L is semisimple as it satisfies (a).

In the fifth example we realise a semisimple LSA following a procedure slightly different from the one prescribed by the main theorem. However this LSA is in fact isomorphic to one found in example 1; this confirms the point (b) of the main theorem.

Example 1. We consider $S = S_1 \otimes \Lambda(n_1) = \mathrm{sl}(1/2) \otimes \Lambda(2)$.

(i) $\mathrm{G} = \mathrm{sl}(1/2)$ is an eight-dimensional, rank-2, simple LSA; $\mathrm{G}_0 = \mathrm{sl}(2) + \mathrm{u}(1)$, $\mathrm{G}_1 = (2)_{+1} + (2)_{-1}$ as G_0 modules. We can represent G by 3×3 matrices:

$$\mathrm{G}_0\colon H = \left(\begin{array}{c|cc} 0 & & \\ \hline & 1 & \\ & & -1 \end{array}\right) \qquad Q = \left(\begin{array}{c|cc} 2 & & \\ \hline & 1 & \\ & & 1 \end{array}\right)$$

$$E_+ = \left(\begin{array}{c|cc} 0 & & \\ \hline & 0 & 1 \\ & & 0 \end{array}\right) \qquad E_- = \left(\begin{array}{c|cc} 0 & & \\ \hline & 0 & \\ & 1 & 0 \end{array}\right)$$

$$\mathrm{G}_1\colon F_{+1} = \left(\begin{array}{c|cc} 0 & 1 & \\ \hline & 0 & \\ & & 0 \end{array}\right) \qquad F_{+2} = \left(\begin{array}{c|cc} 0 & & 1 \\ \hline & 0 & \\ & & 0 \end{array}\right)$$

$$F_{-1} = \left(\begin{array}{c|cc} 0 & & \\ \hline 1 & 0 & \\ & & 0 \end{array}\right) \qquad F_{-2} = \left(\begin{array}{c|cc} 0 & & \\ \hline & 0 & \\ 1 & & 0 \end{array}\right).$$

Remarks. H, E_+, E_- are the generators of the sl(2), Q is the one of the u(1), it verifies the commutation relations $[Q, X] = 0$, $\forall X \in \mathrm{G}_0$ and $[Q, F_{\pm i}] = \pm F_{\pm i}$, $i = 1, 2$,

$$\operatorname{der} \mathrm{sl}(1/2) \approx \mathrm{sl}(1/2).$$

(ii) $\Lambda(2)$ is a four-dimensional associative superalgebra with basis vectors 1, ξ_1, ξ_2, $\xi_1\xi_2$ where the ξ_i are Grassmann variables. A generator $X \otimes u$ is represented by the 3×3 matrix representing X, but where every non-zero entries are multiplied by u, for example:

$$\left(\begin{array}{c|cc} 0 & & \\ \hline & \xi_1 & \\ & & -\xi_1 \end{array}\right) \quad \left(\begin{array}{c|cc} 0 & & \\ \hline & 0 & \xi_1\xi_2 \\ & & 0 \end{array}\right) \quad \left(\begin{array}{c|cc} 0 & & \\ \hline 1 & 0 & \\ & & 0 \end{array}\right) \quad \left(\begin{array}{c|cc} 0 & & \xi_2 \\ \hline & 0 & \\ & & 0 \end{array}\right)$$

and so on. Let $\mathrm{G} \cdot u$ denote the space spanned by the set of $X \otimes u$, $\forall X \in \mathrm{G}$, for a given $u \in \Lambda(2)$; then we have the following graph of inder S:

$$\begin{tikzcd} & \mathrm{G} \arrow[dl] \arrow[dr] \arrow[dd] & \\ \mathrm{G} \cdot \xi_1 \arrow[dr] & & \mathrm{G} \cdot \xi_2 . \arrow[dl] \\ & \mathrm{G} \cdot \xi_1\xi_2 & \end{tikzcd}$$

To be more complete, consider the non-trivial ideals of $\Lambda(2)$:

$$I_1\colon \langle \xi_1;\ \xi_2;\ \xi_1\xi_2 \rangle \qquad I_2\colon \langle \xi_1;\ \xi_1\xi_2 \rangle \qquad I_3\colon \langle \xi_2;\ \xi_1\xi_2 \rangle$$

$$I_4\colon \langle \xi_1\xi_2 \rangle \qquad I_5(\lambda)\colon \langle \xi_1 + \lambda\xi_2;\ \xi_1\xi_2 \rangle \qquad 0 \neq \lambda \neq \infty.$$

Let F_i denote the factor space $\Lambda(2)/I_i$; $\mathrm{G}\langle F_i \rangle$ and $\mathrm{G}\langle I_i \rangle$ are spanned by $\{X \otimes u$ such that $X \in \mathrm{G}$; $u \in F_i$, respectively $u \in I_i\}$, then the following graph of inder S shows that this last LSA is not semisimple since $[\ldots[G\langle I_i\rangle, G\langle I_i\rangle\}, \ldots, G\langle I_i\rangle\} = 0$:

$$\begin{array}{c} G\langle F_i\rangle \\ \big\downarrow \\ G\langle I_i\rangle. \end{array}$$

Consider now the following commutation relation:

$$\Bigg[\underbrace{\begin{pmatrix} \xi_1\partial_{\xi_2} & 0 & 0 \\ 0 & \xi_1\partial_{\xi_2} & 0 \\ 0 & 0 & \xi_1\partial_{\xi_2} \end{pmatrix}}_{\in 1\otimes \operatorname{der}\Lambda(2)}, \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{pmatrix}}_{\in G\langle I_3\rangle}\Bigg] = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi_1 \\ 0 & 0 & 0 \end{pmatrix}}_{\in G\langle F_3\rangle}.$$

This example provides a good illustration that

(1) inder S is an ideal of L,

(2) due to the element $1\otimes\xi_1\partial_{\xi_2}$ we obtain the following subgraph of L, for $i=3$:

$$\begin{array}{c} G\langle F_i\rangle \\ \big\updownarrow \\ G\langle I_i\rangle. \end{array}$$

In fact, in order to obtain semisimplicity we need subgraphs like the above one, $\forall(F_i, I_i)$. Therefore, in order that no $G\langle I_i\rangle$ remains invariant, the use of a suitable subset of derivations $1\otimes D$ in $1\otimes\operatorname{der}\Lambda(2)$ is needed (hence the formulation of point (a)); this will now be done.

(iii) Let the eight basis vectors of $\operatorname{der}\Lambda(2)\approx W(2)$ be

$$\{D_1, \ldots, D_8\} \equiv \{\partial_{\xi_1}; \partial_{\xi_2}; \xi_1\partial_{\xi_1}; \xi_2\partial_{\xi_1}; \xi_1\partial_{\xi_2}; \xi_2\partial_{\xi_2}; \xi_1\xi_2\partial_{\xi_1}; \xi_1\xi_2\partial_{\xi_2}\}.$$

Due to the above-mentioned commutation relations $[1\otimes D_i, X\otimes u\} \equiv X\otimes D_i(u)$ we see that the non-trivial part of the job is to compute the $D_i(u)$.

It is easy to check if $\exists u\in I_j$ such that $D_i(u) = v\notin I_j$. In that case D_i does not leave I_j invariant and we note this by a Y in table 1. Otherwise if D_i does leave I_j invariant we note this by a point.

We really need the two derivations D_1 and D_2 since the derivation $D = a\partial_{\xi_1} + b\partial_{\xi_2}$, $a\neq 0\neq b$ leaves invariant the ideal $I_5(-a/b)$.

Table 1.

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
I_1	Y	Y	.	.	.	.	.	.
I_2	Y	.	.	Y	.	.	.	.
I_3	.	Y	.	.	Y	.	.	.
I_4	Y	Y	.	.	.	.	.	.
I_5	Y	Y	Y	Y	Y	Y	.	.

Proof. Any $V \in I_5(\lambda)$ can be written $V = v_1(\xi_1 + \lambda\xi_2) + v_2\xi_1\xi_2$, then $DV = v_1(a + b\lambda) - v_2 \cdot b(\xi_1 + \lambda\xi_2) + v_2(a + b\lambda)\xi_2$.

(iv) Let $L_1 = \langle D_1;\, D_2\rangle$ denote the space spanned by the derivations D_1 and D_2. Then $L = \text{inder}\,\mathrm{S} \oplus L_1$ is clearly a semisimple LSA: this is our first example. The graph of L is

$$
\begin{tikzcd}
& \mathrm{sl}(1/2)\cdot\xi_1 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & & \\
\mathrm{sl}(1/2)\cdot\xi_1\xi_2 \arrow[dr, leftrightarrow] & & \mathrm{sl}(1/2) \arrow[ll] \arrow[dl, leftrightarrow] & \langle D_1;\, D_2\rangle. \arrow[l] \arrow[ull] \arrow[dll] \\
& \mathrm{sl}(1/2)\cdot\xi_2 & &
\end{tikzcd}
$$

Another semisimple LSA is $L = \text{inder}\,\mathrm{S} \oplus \langle D_1;\, D_2;\, D_4\rangle$, the graph of this second example is

$$
\begin{tikzcd}
& \mathrm{sl}(1/2)\cdot\xi_1 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[dd] & & \langle D_2 = \partial_{\xi_2}\rangle \arrow[lll] \arrow[dl] \arrow[d] \\
\mathrm{sl}(1/2)\cdot\xi_1\xi_2 \arrow[dr, leftrightarrow] & & \mathrm{sl}(1/2) \arrow[ll] \arrow[dl, leftrightarrow] & \langle D_1 = \partial_{\xi_1}\rangle \arrow[l] \arrow[dll] \\
& \mathrm{sl}(1/2)\cdot\xi_2 & & \langle D_4 = \xi_2\partial_{\xi_1}\rangle. \arrow[ll] \arrow[u]
\end{tikzcd}
$$

(v) Note that in general $\text{inder}\,\mathrm{S} + M$, where M is spanned by a set of derivations that do not leave all the ideals of $\Lambda(2)$ invariant, is not a LSA, as there can exist some derivations in $[M, M\}$ that do not belong to M. Therefore we have to take $\mathrm{L}_1 = M + [M, M\} + [[M, M\}, M\} + \ldots$, in order for $\mathrm{L} = \text{inder}\,\mathrm{S} \oplus \mathrm{L}_1$ to be a Lie superalgebra.

For example, we cannot have $\mathrm{L}_1 = \langle d_1, d_2\rangle$ where $d_1 \equiv D_1 - D_7$ and $d_2 \equiv D_2 - D_8$, because $\frac{1}{2}\{d_1, d_1\} = D_4 \equiv d_3$, $\{d_1, d_2\} = D_3 - D_6 \equiv d_4$ and $\frac{1}{2}\{d_2, d_3\} = D_5 \equiv d_5$. Note that the two derivatives D_1 and D_2 do appear only as components of d_1 and d_2 and that d_1, d_2, d_3, d_4 and d_5 are the five generators of the simple LSA called $\tilde{\mathrm{S}}(2)$, which is isomorphic to $\mathrm{osp}(1/2)$. This gives a third example of semisimple LSA: $\text{inder}\,\mathrm{S} \oplus \tilde{\mathrm{S}}(2)$.

Other examples, D_1, D_2, D_4, D_5 and $D' \equiv D_3 - D_6$, are the five generators of the non-semisimple LSA called $\mathrm{S}(2)$ ($\mathrm{S}(n)$ is simple for $n \geqslant 3$). The fourth example of semisimple LSA is then $\mathrm{L} = \text{inder}\,\mathrm{S} \oplus \mathrm{S}(2)$.

The graphs of these last two semisimple LSA are respectively

$\underline{\mathrm{sl}(1/2) \otimes \Lambda(2) \oplus \tilde{\mathrm{S}}(2)}$

$$
\begin{tikzcd}
& \mathrm{sl}(1/2)\cdot\xi_2 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[dd, leftrightarrow] & & \\
\mathrm{sl}(1/2)\cdot\xi_1\xi_2 \arrow[dr, leftrightarrow] & & \mathrm{sl}(1/2) \arrow[ll] \arrow[dl, leftrightarrow] & \tilde{S}(2)_0 \arrow[l] \arrow[d, leftrightarrow] \\
& \mathrm{sl}(1/2)\cdot\xi_2 & & \tilde{\mathrm{S}}(2)_1
\end{tikzcd}
$$

$\underline{\mathrm{sl}(1/2) \otimes \Lambda(2) \oplus \mathrm{S}(2)}$

$$
\begin{tikzcd}
& \mathrm{sl}(1/2)\cdot\xi_1 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[dd, leftrightarrow] & & \\
\mathrm{sl}(1/2)\cdot\xi_1\xi_2 \arrow[dr, leftrightarrow] & & \mathrm{sl}(1/2) \arrow[ll] \arrow[dl, leftrightarrow] & S(2)_0 \arrow[l] \arrow[d] \\
& \mathrm{sl}(1/2)\cdot\xi_2 & & \mathrm{S}(2)_1
\end{tikzcd}
$$

where $\tilde{\mathrm{S}}(2) = \tilde{\mathrm{S}}(2)_0 + \tilde{\mathrm{S}}(2)_1$ and $\mathrm{S}(2) = \mathrm{S}(2)_0 + \mathrm{S}(2)_1$.

(vi) According to point (c) of the main theorem and using the following results: $\text{der}\,\tilde{\mathrm{S}}(2) \approx \tilde{\mathrm{S}}(2)$ and $\text{der}\,\mathrm{S}(2) \approx \mathrm{S}(2)^z$, where the meaning of $\mathrm{S}(2)^z$ is given in the appendix, we obtain $\text{der}\,\mathrm{L} \approx \mathrm{L}$ if $\mathrm{L} \approx \mathrm{sl}(1/2) \otimes \Lambda(2) \oplus \tilde{\mathrm{S}}(2)$ but $\text{der}\,\mathrm{L} \approx \mathrm{L}^z$ where L^z denotes $\mathrm{sl}(1/2) \otimes \Lambda(2) \oplus \mathrm{S}(2)^z$ if $\mathrm{L} \approx \mathrm{sl}(1/2) \otimes \Lambda(2) \oplus \mathrm{S}(2)$.

Example 2. We consider now $S = G \otimes \Lambda(1)$, where G is a simple LSA.

(i) $\Lambda(1)$ is two dimensional with basis vectors 1, ξ. The only non-trivial ideal J is obviously spanned by ξ. $W(1)$ is a solvable two-dimensional LSA with basis vectors ∂_ξ and $\xi\partial_\xi$. As only the derivation ∂_ξ does not leave J invariant, the possible L_1 are the spaces $L_1' = \langle\partial_\xi\rangle$ and $L_1'' = \langle\partial_\xi, \xi\partial_\xi\rangle$.

The graph of the LSA $K = \text{inder}\, S + \langle\xi\partial_\xi\rangle$ indicates clearly why $\langle\xi\partial_\xi\rangle$ cannot span a good L_1:

$$\begin{tikzcd} G \arrow[d] & \\ G\cdot\xi & \langle\xi\partial_\xi\rangle \arrow[l] \end{tikzcd}.$$

(ii) If $\text{der}\, G \approx G$, then the two semisimple LSA we can obtain the following graphs:

$$\begin{tikzcd} G \arrow[d, leftrightarrow] & \langle\partial_\xi\rangle \arrow[l] \\ G\cdot\xi & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[d, leftrightarrow] & \langle\partial_\xi\rangle \arrow[l] \\ G\cdot\xi & \langle\xi\partial_\xi\rangle \arrow[l] \arrow[u] \end{tikzcd}.$$

We call these two semisimple LSA (E2.1) and (E2.2).

(iii) If $\text{der}\, G \approx G \oplus \langle z\rangle \equiv G^z$ where $\langle z\rangle$ is a one-dimensional space spanned by the derivation z, the graphs of the six possible semisimple LSA, that we name from (E2.3) to (E2.8), are

$$\begin{tikzcd} G \arrow[d, leftrightarrow] & \langle\partial_\xi\rangle \arrow[l] \\ G\cdot\xi & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[d, leftrightarrow] & \langle\partial_\xi\rangle \arrow[l] \\ G\cdot\xi & \langle\xi\partial_\xi\rangle \arrow[l] \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} \langle z\rangle \arrow[r] \arrow[dr] & G \arrow[d, leftrightarrow] & \langle\partial_\xi\rangle \arrow[l] \\ & G\cdot\xi & \end{tikzcd} \qquad \begin{tikzcd} \langle z\rangle \arrow[r] \arrow[dr] & G \arrow[d, leftrightarrow] & \langle\partial_\xi\rangle \arrow[l] \\ & G\cdot\xi & \langle\xi\partial_\xi\rangle \arrow[l] \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} \langle z\rangle \arrow[r] \arrow[dr] & G \arrow[d, leftrightarrow] & \langle\partial_\xi\rangle \arrow[l] \arrow[ll, bend right] \\ \langle z\otimes\xi\rangle \arrow[u] \arrow[r] & G\cdot\xi & \end{tikzcd} \qquad \begin{tikzcd} \langle z\rangle \arrow[r] \arrow[dr] & G \arrow[d, leftrightarrow] & \langle\partial_\xi\rangle \arrow[l] \arrow[ll, bend right] \\ \langle z\otimes\xi\rangle \arrow[u] \arrow[r] & G\cdot\xi & \langle\xi\partial_\xi\rangle \arrow[l] \arrow[u] \arrow[ll, bend left] \end{tikzcd}.$$

Note that the following vector spaces $\langle G; G\cdot\xi; z\cdot\xi; \partial_\xi\rangle$ and $\langle G; G\cdot\xi; z\cdot\xi; \partial_\xi; \xi\partial_\xi\rangle$ are not Lie superalgebras as $\{z\otimes\xi, \partial_\xi\} = z$:

$$\begin{tikzcd} \bullet & G \arrow[d, leftrightarrow] & \langle\partial_\xi\rangle \arrow[l] \arrow[ll, bend right, dashed] \\ \langle z\otimes\xi\rangle \arrow[u, dashed] \arrow[r] & G\otimes\xi & \end{tikzcd} \qquad \begin{tikzcd} \bullet & G \arrow[d, leftrightarrow] & \langle\partial_\xi\rangle \arrow[l] \arrow[ll, bend right, dashed] \\ \langle z\cdot\xi\rangle \arrow[u, dashed] \arrow[r] & G\cdot\xi & \langle\xi\partial_\xi\rangle \arrow[l] \arrow[u, dashed] \arrow[ll, bend left] \end{tikzcd}.$$

(iv) We simply have

$$\begin{array}{ll} \operatorname{der}(E2.i) \approx \operatorname{der}(E2.2) & i = 1, 2 \\ \operatorname{der}(E2.j) \approx \operatorname{der}(E2.6) & j = 3, 4, 6 \\ \operatorname{der}(E.2k) \approx \operatorname{der}(E2.8) & k = 5, 7, 8. \end{array}$$

Remark. Let H be a simple LA. Then $L = H \otimes \Lambda(1) + \langle \partial_\xi \rangle$ is called H^ξ in [1]. By extension, if in place of H we have a simple LSA say G, we will call G^ξ the corresponding L.

Example 3. We consider now $S = \bigoplus_i S_i \otimes \Lambda(n_i)$ where each n_i is zero.

(i) In that case $L = \bigoplus_i S_i$ is a direct sum of simple LSA.

(ii) $\operatorname{der} L = \bigoplus_i \operatorname{der} S_i$.

Example 4. Let $L = \bigoplus_{i=1}^r (S_i \otimes \Lambda(n_i) + \langle \partial_{\xi_i} \rangle)$ where all the n_i are equal to 1. It is a generalisation of example 2 where r was put to 1.

Strictly speaking it describes a direct sum of the S^ξ type of semisimple LSA. Thus its graph is

$$\begin{array}{l} S_1 \cdot \xi_1 \leftrightarrow S_1 \leftarrow \langle \partial_{\xi_1} \rangle \\ S_2 \cdot \xi_2 \leftrightarrow S_2 \leftarrow \langle \partial_{\xi_2} \rangle \\ \qquad\quad \vdots \\ S_r \cdot \xi_r \leftrightarrow S_r \leftarrow \langle \partial_{\xi_r} \rangle. \end{array}$$

We have also the implicit possibility to identify all the $\Lambda(1)$ giving

$$\begin{tikzcd}[row sep=small] S_1 \cdot \xi \leftrightarrow S_1 & \\ S_2 \cdot \xi \leftrightarrow S_2 & \langle \partial_\xi \rangle. \arrow[ul] \arrow[l] \arrow[dl] \\ \vdots & \\ S_r \cdot \xi \leftrightarrow S_r & \end{tikzcd}$$

The possible identification of all the $\Lambda(n_i)$ is in fact foreseen in the formulation of the main theorem; it is sufficient to see the graph of the previous LSA in the following way:

$$\begin{tikzcd}[row sep=small] S_1 \cdot \xi_1 \leftrightarrow S_1 & \\ S_2 \cdot \xi_2 \leftrightarrow S_2 & \langle a_1 \partial_{\xi_1} + a_2 \partial_{\xi_2} + \ldots + a_r \partial_{\xi_r} \rangle \arrow[ul] \arrow[l] \arrow[dl] \\ \vdots & \\ S_r \cdot \xi_r \leftrightarrow S_r & \end{tikzcd} \qquad a_i = 0 \ \forall i = 1, \ldots, r.$$

(L_i, the component of L in $1 \otimes \operatorname{der} \Lambda(n_i)$, is given by $a_i \partial_{\xi_i}$. If some $a_i = 0$, then L is not semisimple.)

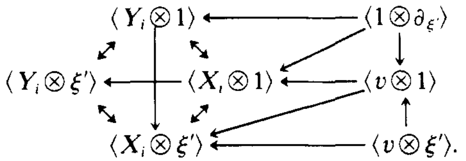
This LSA is called $G(S_1, \ldots, S_r, \langle a_1 \partial_{\xi_1} + a_2 \partial_{\xi_2} + \ldots + a_r \partial_{\xi_r} \rangle)$ by Kac [1b]. The generalisation of this last notation is clear, for example, $G(S_1, S_2, S_3, \langle \partial_{\xi_1}; a\partial_{\xi_2} + b\partial_{\xi_3} \rangle)$ has the following graph:

$$\begin{tikzcd}[row sep=small] S_1 \cdot \xi_1 \leftrightarrow S_1 & \langle \partial_{\xi_1} \rangle \arrow[l] \\ S_2 \cdot \xi_2 \leftrightarrow S_2 & \\ S_3 \cdot \xi_3 \leftrightarrow S_3 & \langle a\partial_{\xi_2} + b\partial_{\xi_3} \rangle. \arrow[ul] \arrow[l] \end{tikzcd}$$

Example 5. Let G be a simple LSA and its generators x_i satisfy the following commutation relations: $[x_i, x_j\} = f_{ij}^k x_k$. If we identify the elements of the LSA G^ξ in the following way: $X_i \equiv x_i \otimes 1$, $Y_i \equiv y_i \otimes \xi$ and $v \equiv 1 \otimes \partial_\xi$, then

(1) $$\begin{array}{ll} [X_i, X_j\} = f_{ij}^k X_k & [v, X_i\} = 0 \\ [X_i, Y_j\} = f_{ij}^k Y_k & [v, Y_j\} = X_i \\ [Y_i, Y_j\} = 0 & \{v, v\} = 0. \end{array}$$

(2) $T \equiv G^\xi \otimes \Lambda(1) + \langle \partial_{\xi'} \rangle$ is a semisimple LSA, whose graph is



T is realised in a way which is not exactly given by the main theorem. We see however that T has the same graph as the second semisimple LSA appearing in example 1; furthermore they really are isomorphic. This confirms point (b) of the main theorem.

5. The infinite case

5.1. The Lie algebra case

The Lie algebras are trivially graded Lie superalgebras. Therefore it is also possible, using the main theorem, to build semisimple LA which are not direct sums of simple ones. Instead of taking the tensor product of a semisimple Lie algebra with a Grassmann algebra, we take it with a commuting algebra, the polynomial ring $K[x_1, \ldots, x_n]$ for example.

The commuting nature of these algebras implies, however, the corresponding semisimple Lie algebras to be infinite.

Example. If we take $S = G \otimes K[x]$, G being a simple finite Lie algebra, we can now consider some Lie algebras L such that

$$\text{inder } S \equiv S \subset L \subset \text{der } S \equiv S \oplus \text{der } K[x]$$

where $K_1 \equiv \text{der } K[x]$ is the well known contact Lie algebra whose set of basis vectors can be realised as $\{x^i \cdot \partial_x, \forall i \in Z_+ U\{0\}\}$.

By analogy with the notation of a previous remark (§ 3), we call G^x the following Lie algebra: $G \otimes K[x] + \langle \partial_x \rangle$.

If we consider the loop algebra $S = G \otimes C[x, x^{-1}]$, where G is a semisimple LA and $C[x, x^{-1}]$ is the Laurent polynomial algebra, we can also obtain some semisimple Lie algebra L where $S \subset L \subset \text{der } S$.

Note that $\text{der } S = S \oplus V$. ($V \equiv \text{der } C[x, x^{-1}]$ is the well known centreless Virasoro algebra.)

Remark 1. The generators u_i of the Virasoro algebra, which satisfy in their canonical form the commutation relations

$$[u_i, u_j] = (i - j) \cdot u_{i+j} \qquad \forall i, j \in Z$$

can be realised in the following way: $u_i \equiv x^{i+1} \cdot \partial_x$. This algebra can also be realised by gluing carefully two copies of the above-mentioned Lie algebra K_1 [4].

Remark 2. The loop algebra $L(G) \equiv G \otimes C[x, x^{-1}]$, where G is a simple finite LA, is also a simple (infinite) LA. However it is the non-semisimple algebra $\hat{L}(G) \equiv L(G) \dotplus \langle u_0 \rangle \oplus \langle c \rangle$, where $\langle c \rangle$ denotes the centre, which has the nice algebraic property of being defined by the means of canonical generators and a Cartan matrix. In fact $\hat{L}(G)$ is one of the Kac-Moody algebras, of the affine non-twisted type.

5.2. The Lie superalgebra case

An extension of § 5.1, where G are taken to be finite simple LSA, is trivial. (Note however that $\hat{L}(G)$ is a Kac-Moody superalgebra only if the Killing form of G is non-degenerate.)

Example 1. Let $S = G \otimes C[x, x^{-1}]$ where G is a simple LSA. Then $\operatorname{der} S = S \dotplus V$, where V is again the centreless Virasoro algebra, is an infinite semisimple LSA.

Note that the Neveu-Schwarz and Ramond LSA (which are the two standard LSA built around the Virasoro algebra) do not appear when considering the LSA of the derivations of a loop superalgebra (i.e. the LSA built around a loop algebra):

$$V \approx \operatorname{der} S/S \qquad \text{where S is a loop (super)algebra.}$$

A little more difficult is the case when considering the tensor product of G with the superalgebras $\Lambda(m, n) \equiv K[x_1, \ldots, x_m] \otimes \Lambda(n)$; happily the structure of $W(m, n) \equiv \operatorname{der} \Lambda(m, n)$, and some of its subalgebras generalising the Cartan simple Lie algebras, is sufficiently well known [1].

Example 2. Consider $S = G \otimes \Lambda(1, 1)$ where G is a finite simple LSA.

A base of $\Lambda(1, 1)$ is given by $\{x^m, \xi \cdot x^n\}$. Correspondingly $\{x^i \cdot \partial_x;\ \xi x^j \cdot \partial_x;\ x^m \cdot \partial_\xi;\ \xi x^n \cdot \partial_\xi\}$ is a base of $W(1, 1)$.

The important subalgebra $K(1, 1)$ of $W(1, 1)$ is spanned by the derivations satisfying

$$\left(\frac{\partial a}{\partial x} - \frac{\partial a}{\partial \xi}\right)\partial_\xi + \left((-1)^{\varepsilon(a)} \cdot 2a + \frac{\partial a}{\partial \xi} \cdot \xi\right)\partial_x \qquad \forall a \in \Lambda(1, 1).$$

An explicit base of $K(1, 1)$ is given by

$$\{D_i \equiv (i x^{i-1} \xi \partial_\xi + 2x^i \cdot \partial_x);\ d_j \equiv x^j(\partial_\xi + \xi \cdot \partial_x)\}.$$

These derivations satisfy the following commutation relations:

$$[D_i, D_j] = 2(i - j) \cdot D_{i+j-1}$$
$$[D_i, d_j] = (i - 2j) \cdot d_{i+j-1}$$
$$\{d_i, d_j\} = D_{i+j}$$

as we can easily check by applying these derivations on any function $f(x, \xi) = a(x) \cdot \xi + b(x)$. Again $L = S \dotplus 1 \otimes K(1, 1)$ is semisimple.

If we extend the previous example to the negative powers of x, then $S = L(G) + L(G) \cdot \xi$ and we obtain a semisimple LSA which has the following graph:

$$\begin{tikzcd} L(G) \arrow[dd, leftrightarrow] & \\ & NS \arrow[ul] \arrow[dl] \\ L(G) \cdot \xi & \end{tikzcd}$$

where NS denotes the centreless Neveu-Schwarz superalgebra.

Proof. First, the structure of the '*extended* (by $\Lambda(1)$) loop superalgebra' $L(G)+L(G)\cdot\xi \equiv G\otimes C(x, x^{-1})\otimes\Lambda(1)$simply comes from the definitions; next, the canonical commutation relations among the generators of $NS\equiv\langle u_i;\, v_j\rangle$, $i\in Z, j\in Z+\frac{1}{2}$, which are

$$[u_i, u_j]=(i-j)\cdot u_{i+j}$$

$$[u_i, v_j]=(\tfrac{1}{2}i-j)\cdot v_{i+j}$$

$$\{v_i, v_j\}=2u_{i+j}$$

can be recovered if we perform the following identification:

$$u_i=\tfrac{1}{2}D_{i+1}$$

$$v_j=d_{j+1/2}.$$

Keeping these definitions for u_i and v_j, but both with $i, j\in Z$, gives a realisation of the Ramond superalgebra R. The action of these derivations on an extended loop superalgebra can be computed if we realise this last one in the following way:

$$G\otimes C(x^{1/2}, x^{-1/2})\otimes\Lambda(1).$$

Note that the centreless NS and R do appear as subalgebras of der S/S, where S is an '*extended*' loop (superalgebra).

The unitary irreducible (positive energy) representations of the central extension of many of these LSA and their applications to two-dimensional quantum theory are discussed in a recent paper [6] of Kac and Todorov. We compare the respective notation below.

Present paper	Kac–Todorov
G	dG
$S=L(G)\otimes\Lambda(1)$	$\widetilde{dG}=dG\otimes C[t, t^{-1};\,\theta]$; $\deg(t)=1$
$S\dotplus\langle c\rangle$ with $\begin{cases} C(t, t^{-1})\otimes\Lambda(1) \\ C(t^{1/2}, t^{-1/2})\otimes\Lambda(1)\end{cases}$	$(\widehat{dG})_{1/2}=\widetilde{dG}+\langle c\rangle$; $\deg\theta=\frac{1}{2}$ $(\widehat{dG})_0=\widetilde{dG}+\langle c\rangle$, $\deg\theta=0$
$V\dotplus\langle c\rangle$	V
$NS\dotplus\langle c\rangle$	$SV_{1/2}$
$R\dotplus\langle c\rangle$	SV_0
$S\dotplus NS\dotplus\langle c\rangle$	$S_{1/2}(G)=(\widehat{dG})_{1/2}+SV_{1/2}$
$S\dotplus R\dotplus\langle c\rangle$	$S_0(G)=(\widehat{dG})_0+SV_0$.

The degree of t and θ refers to the Z gradation of the $Z\otimes Z_2$ gradation of these LSA.

Acknowledgment

I am grateful to Professor H Ruegg for helpful advice.

Appendix

The LSA of the derivations of the simple LSA which are listed in table A1 are described in references [1, 2]. Readers are warned that in this table we retain the notation of

Table A1.

G	Killing form	der G	dim(der G) − dim(G)
$A(m, n)\ m \neq n$	*	G	0
$A(n, n)\ n \neq 1$		G^z	1
$A(1, 1)$		$D(2, 1, \alpha = 0)$	3
$C(n)$	*	G	0
$B(m, n)$	*	G	0
$D(m, n)\ m \neq n+1$	*	G	0
$D(n+1, n)$		G	0
$D(2, 1, \alpha)\alpha \neq 0, -1, \infty.$		G	0
$F(4)$	*	G	0
$G(3)$	*	G	0
$P(n)$		G^z	1
$Q(n)$		der $Q(n)$	1
$W(n)$		G	0
$S(n)$		G^z	1
$\tilde{S}(n)$		G	0
$H(n)$		$\tilde{H}(n)^z$	2

[1]; in particular a superalgebra $G^Z \equiv G \oplus \langle z \rangle$ denotes the (finite) LSA for which the following commutation relations hold: if $G = \bigoplus_k G_k$, then $[z, X] = k \cdot X$, $\forall X \in G_k$: it has nothing to do with the infinite Lie (super)algebra G^X defined in § 4.

References

[1a] Kac V G 1977 *Adv. Math.* **26** 8-96

[1b] Kac V G 1979 *Lecture Notes in Mathematics* vol 716 (Berlin: Springer)

[3] Wess J and Bagger J *Supersymmetry and Supergravity* (*Princeton Series in Physics*) (Princeton, NJ: Princeton University Press)

[4] Kaplansky I 1982 *Commun. Math. Phys.* **86** 49-54

[5] Djokovic D Z and Hochschild G 1976 *J. Math.* **20** 134

[6] Kac V G and Todorov I T 1985 *Commun. Math. Phys.* **102** 337